Published by Institute of Physics Publishing for SISSA/ISAS
Received: *November 4, 2004*
Revised: *December 6, 2004*
Accepted: *December 6, 2004*

Colliding plane wave solutions in string theory revisited

Bin Chen

Interdisciplinary Center of Theoretical Studies, Chinese Academy of Science
P.O. Box 2735, Beijing 100080, P.R. China
E-mail: bchen@itp.ac.cn

Abstract: We construct the colliding plane wave solutions in the higher-dimensional gravity theory with fluxes and dilaton, with a more general ansatz on the metric. We consider two classes of solutions to the equations of motions and after imposing the junction conditions we find that they are all physically acceptable. In particular, we manage to obtain the higher-dimensional Bell-Szekeres solutions in the Maxwell-Einstein gravity theory, and the flux-CPW solutions in the eleven-dimensional supergravity theory. All the solutions have been shown to develop the late time curvature singularity.

Keywords: Superstrings and Heterotic Strings, Classical Theories of Gravity.

Contents

1. Introduction

The colliding plane wave solutions (CPW) have been an important topic in the classical general relativity. They are the exact solutions describing the collision of plane wave in a flat background. In the four-dimensional gravity theory, the study of CPW solutions started from the early 1970s [1, 2]. Since then, many exact CPW solutions have been constructed. For a thorough review on the four dimensional CPW solutions, see [3]. One remarkable feature of CPW solutions in four dimension is that they always develop a late time curvature singularity [4, 5]. This fact could be considered as an inevitable effect of the nonlinear gravitational focusing. However, the initial background could play an essential role too. In [6], it has been shown that in the case that the initial background is expanding, the focusing effect of the nonlinear interaction between the waves could be weakened by the expansion so that no future singularity occurs. It has been expected that the study of the CPW solutions may tell us the nature of the space-time singularity. Also, it has been proposed that the gravitational plane and dilatonic waves could play an important role in the pre-big-bang cosmology scenarios [7, 8].

It is very interesting to study the colliding plane wave solutions in the higher dimensional gravity theory. One reason is that we may live in a higher dimensional spacetime and we have string/M theory in 10D/11D as a candidate to describe the world. The higher dimensional gravity theory with dilaton and various fluxes could be taken as the low energy effective action of the string theory. It has manifested some different features from the four dimensional gravity theory. For example, it has been found that the uniqueness and stability issue in higher dimensional black holes are more subtle [9] and there exist the *black ring* solution with horizon topology $S^1 \times S^2$ in five dimensional gravity [10]. Therefore, we can wish that the study of CPW solutions reveal some new features of the higher-dimensional gravity theory. On the other hand, due to the existence of the dilaton and the various fluxes, the CPW solutions in the theory have richer structure. There have been some efforts in this direction. In [11, 12, 13], the CPW solutions in the dilatonic gravity, in the higher dimensional gravity, and in the higher dimensional Einstein-Maxwell theory have been discussed. In [14], Gutperle and Pioline tried to construct the CPW solutions in the ten-dimensional gravity with the self-dual form flux. However, their solutions failed to satisfy the junction condition and are physically unacceptable.

To study the collinear CPW solutions in the higher dimensional gravity, one could make a quite general ansatz

$$ds^2 = 2e^{-M}dudv + \sum_{i=1}^{k} e^{A_i}dx_i^2, \qquad (k \geq 3) \tag{1.1}$$

where M, A_i are only the function of u, v. Obviously, with such a generic ansatz, it will be very difficult to solve the equations of motions. In [16, 19], we have assumed $A = A_i$, $i = 1, \dots, n(n < k)$ and $B = A_i$, $i = n+1, \dots, k$ so that

$$ds^2 = 2e^{-M}dudv + e^A \sum_{i=1}^{n} dx_i^2 + e^B \sum_{j=n+1}^{k} dy_j^2 \tag{1.2}$$

to simplify the equations and tried to find the flux-CPW solutions with dilaton. In [16], we managed to find two classes of 1-flux-CPW solutions satisfying the junction conditions: $(pqrw)$-type and $(f \pm g)$-type. The $(pqrw)$-type solutions look like the Bell-Szekeres solution, which appeared in four-dimensional Einstein-Maxwell gravity [15]. The $(f \pm g)$ type solutions is a new kind of solutions. Unfortunately, we also noticed that with metric (1.2), it is impossible to have well-defined CPW solutions if we turn off the dilaton. More precisely, we found no higher-dimensional BS solution in Maxwell-Einstein gravity and no pure flux solution in eleven dimensional supergravity which has no dilaton. In other words, the only pure flux-CPW solution could only be four-dimensional BS solution. Nevertheless, with dilaton turning on, there always exist well-defined physical flux-CPW solution. In [19], we generalized the discussions on flux-CPW solutions to the case with two complementary fluxes. We found that the $(pqrw)$-type solution is still physically well-defined but $(f \pm g)$-type solution failed to satisfy the junction conditions.

In this paper, in order to investigate the flux-CPW solutions more carefully, we go a little further and try a more complicated ansatz:

$$ds^2 = 2e^{-M}dudv + e^A \sum_{i=1}^{n} dx_i^2 + e^B \sum_{j=1}^{m} dy_j^2 + e^C \sum_{k=1}^{l} dz_k^2 \,. \tag{1.3}$$

With this ansatz, we will find that the equations of motions are still exactly solvable. And generically we will have physically acceptable solutions after imposing the junction conditions. We will work on a dilatonic gravity theory with two fluxes, which could be reduced to other cases easily. Remarkably, we notice that the higher-dimensional pure 1-flux-CPW solutions do exist, with the above metric ansatz.

We organize the paper as follows. In section 2, we derive the equations of motions of the system and simplify them by changing variables and using (f, g) coordinates. In section 3, we construct the physical CPW solutions taking into account of the junction conditions, and discuss the future singularity issue in these solutions. In section 4, we turn to several interesting special cases. In particular, we consider the one-flux case, which has been probed in [16]. We end the paper with some discussions and conclusions.

2. Equations of motions and their reduction

To keep the problem as generic as possible, we start from a dilatonic gravity theory with two fluxes, whose action take the form

$$S = \int d^D x\sqrt{-g}\left(R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2(n+1)!}e^{a\phi}F^2 - \frac{1}{2(m+1)!}e^{b\phi}G^2\right). \tag{2.1}$$

Here, $D \geq 5$ is the dimension of spacetime, ϕ is the dilaton field with a and b being dilaton coupling constant, and F, G are two fluxes. The action could be obtained from the low energy effective action of string/M theory with specific flux configurations in the Einstein frame. The equations of motions are given by

$$\begin{aligned} R_{\mu\nu} = {} & \partial_\mu\phi\partial_\nu\phi + \frac{1}{2n!}e^{a\phi}\left(F_{\mu\mu_1\ldots\mu_n}F_\nu{}^{\mu_1\ldots\mu_n} - \frac{n}{(n+1)(m+n+l)}g_{\mu\nu}F^2\right) + \\ & + \frac{1}{2m!}e^{b\phi}\left(G_{\mu\nu_1\ldots\nu_m}G_\nu{}^{\nu_1\ldots\nu_m} - \frac{m}{(m+1)(m+n+l)}g_{\mu\nu}G^2\right) \end{aligned} \tag{2.2}$$

$$\partial_\mu\left(\sqrt{-g}e^{a\phi}F^{\mu\mu_1\ldots\mu_n}\right) = 0 \tag{2.3}$$

$$\partial_\nu\left(\sqrt{-g}e^{b\phi}G^{\nu\nu_1\ldots\nu_m}\right) = 0 \tag{2.4}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = \frac{a}{4(n+1)!}e^{a\phi}F^2 + \frac{b}{4(m+1)!}e^{b\phi}G^2 \tag{2.5}$$

Also to keep the problem workable, we assume that the fluxes lie along x_i's and y_j's respectively

$$\begin{aligned} F_{ux_1\ldots x_n} &= D_u \qquad & F_{vx_1\ldots x_n} &= D_v \\ G_{uy_1\ldots y_m} &= E_u \qquad & G_{vy_1\ldots y_m} &= E_v\,. \end{aligned} \tag{2.6}$$

In [19], we consider the two complimentary fluxes which occupy the whole spacetime. Here we have additional transverse dimensions. If we let some fluxes or dilaton field vanish, the system can reduce to some interesting special cases, such as one flux case, pure dilatonic case et.al, which is the subject in section 4. One special case is the gravity coupled 1-flux without dilaton, when we turn off the dilaton and let one flux vanish. In eleven dimension, the study of CPW solutions in such gravity theory may effectively describe the CPW solutions in 11D supergravity. Recall that in the bosonic part of 11D supergravity action, we have no dilaton but we have a $A_3 \wedge F_4 \wedge F_4$ term, where A_3 is the antisymmetric 3-form tensor gauge field and F_4 is its field strength. However, with the above assumption on the flux, the extra term does not contribute to the action and the equations of motions. In this sense, the CPW solutions in 11D supergravity could be effectively studied from the action (2.1) after reduction.

In components, the Einstein equation take the forms

$$\begin{aligned} & nA_{uu} + mB_{uu} + lC_{uu} + M_u(nA_u + mB_u + lC_u) + \frac{1}{2}(nA_u^2 + mB_u^2 + lC_u^2) = \\ & \qquad = -2\phi_u^2 - e^{a\phi - nA} D_u^2 - e^{b\phi - mB} E_u^2 \end{aligned} \tag{2.7}$$

$$\begin{aligned} & nA_{vv} + mB_{vv} + lC_{vv} + M_v(nA_v + mB_v + lC_v) + \frac{1}{2}(nA_v^2 + mB_v^2 + lC_v^2) = \\ & \qquad = -2\phi_v^2 - e^{a\phi - nA} D_v^2 - e^{b\phi - mB} E_v^2 \end{aligned} \tag{2.8}$$

$$\begin{aligned} & -M_{uv} + \frac{n}{2}A_{uv} + \frac{m}{2}B_{uv} + \frac{l}{2}C_{uv} + \frac{1}{4}(nA_uA_v + mB_uB_v + lC_uC_v) = \\ & \qquad = -\phi_u\phi_v + \frac{n-m-l}{2(n+m+l)}e^{a\phi - nA}D_uD_v + \frac{m-n-l}{2(n+m+l)}e^{b\phi - mB}E_uE_v \end{aligned} \tag{2.9}$$

$$\begin{aligned} & 2A_{uv} + nA_uA_v + \frac{m}{2}(A_uB_v + A_vB_u) + \frac{l}{2}(A_uC_v + A_vC_u) = \\ & \qquad = -\frac{2(m+l)}{n+m+l}e^{a\phi - nA}D_uD_v + \frac{2m}{n+m+l}e^{b\phi - mB}E_uE_v \end{aligned} \tag{2.10}$$

$$\begin{aligned} & 2B_{uv} + mB_uB_v + \frac{n}{2}(A_uB_v + A_vB_u) + \frac{l}{2}(B_uC_v + B_vC_u) = \\ & \qquad = \frac{2n}{n+m+l}e^{a\phi - nA}D_uD_v - \frac{2(n+l)}{n+m+l}e^{b\phi - mB}E_uE_v \end{aligned} \tag{2.11}$$

$$\begin{aligned} & 2C_{uv} + lC_uC_v + \frac{n}{2}(A_uC_v + A_vC_u) + \frac{m}{2}(B_uC_v + B_vC_u) = \\ & \qquad = \frac{2n}{n+m+l}e^{a\phi - nA}D_uD_v + \frac{2m}{n+m+l}e^{b\phi - mB}E_uE_v \,. \end{aligned} \tag{2.12}$$

Here, as usual, the equation (2.9) is redundant and will not be needed in the following discussion. The equations of motions for the dilaton and n-form, m-form potential are given by

$$2D_{uv} + \left[a\phi - \frac{1}{2}(nA - mB - lC)\right]_u D_v + \left[a\phi - \frac{1}{2}(nA - mB - lC)\right]_v D_u = 0 \tag{2.13}$$

$$2E_{uv} + \left[b\phi + \frac{1}{2}(nA - mB + lC)\right]_u E_v + \left[b\phi + \frac{1}{2}(nA - mB + lC)\right]_v E_u = 0 \tag{2.14}$$

$$\phi_{uv} + \frac{1}{4}(nA+mB+lC)_u\phi_v + \frac{1}{4}(nA+mB+lC)_v\phi_u = \frac{a}{4}e^{a\phi - nA}D_uD_v + \frac{b}{4}e^{b\phi - mB}E_uE_v \,. \tag{2.15}$$

Let us introduce

$$U = \frac{1}{2}(nA + mB + lC) \qquad V = \frac{1}{2}(nA - mB - lC) \qquad W = \frac{1}{2}(nA - mB + lC) \tag{2.16}$$

to rewrite the above equations. From the equations (2.10), (2.11), (2.12), we have

$$U_{uv} + U_u U_v = 0 \tag{2.17}$$

and

$$V_{uv} + \frac{1}{2}(U_u V_v + U_v V_u) = -\frac{n(m+l)}{m+n+l} e^{a\phi - nA} D_u D_v + \frac{mn}{m+n+l} e^{b\phi - mB} E_u E_v \tag{2.18}$$

$$W_{uv} + \frac{1}{2}(U_u W_v + U_v W_u) = -\frac{nm}{m+n+l} e^{a\phi - nA} D_u D_v + \frac{m(n+l)}{m+n+l} e^{b\phi - mB} E_u E_v \,. \tag{2.19}$$

The eq. (2.17) has the well-known solution

$$U = \log\left[f(u) + g(v)\right], \tag{2.20}$$

where f, g are arbitrary functions, chosen usually to be monotonic functions. One can treat (f, g) as coordinates alternative to (u, v). It turns out that it is more convenient to work in (f, g) coordinates.

Moreover, the equations (2.7) and (2.8) become

$$\begin{aligned} U_{uu} + M_u U_u + \frac{1}{4}\left(\frac{m+n}{mn}U_u^2 + \frac{l+n}{ln}V_u^2 + \frac{m+l}{ml}W_u^2 + \frac{2}{n}U_u V_u - \frac{2}{m}U_u W_u - \frac{2}{l}W_u V_u\right) = \\ = -\phi_u^2 - \frac{1}{2}e^{a\phi - nA} D_u^2 - \frac{1}{2}e^{b\phi - mB} E_u^2 \end{aligned} \tag{2.21}$$

$$\begin{aligned} U_{vv} + M_v U_v + \frac{1}{4}\left(\frac{m+n}{mn}U_v^2 + \frac{l+n}{ln}V_v^2 + \frac{m+l}{ml}W_v^2 + \frac{2}{n}U_v V_v - \frac{2}{m}U_v W_v - \frac{2}{l}W_v V_v\right) = \\ = -\phi_v^2 - \frac{1}{2}e^{a\phi - nA} D_v^2 - \frac{1}{2}e^{b\phi - mB} E_v^2 \end{aligned} \tag{2.22}$$

and the equations (2.13), (2.14) and (2.15) read

$$2D_{uv} + (a\phi - V)_u D_v + (a\phi - V)_v D_u = 0 \tag{2.23}$$

$$2E_{uv} + (b\phi + W)_u E_v + (b\phi + W)_v E_u = 0 \tag{2.24}$$

$$\phi_{uv} + \frac{1}{2}(U_u \phi_v + U_v \phi_u) = \frac{a}{4} e^{a\phi - U - V} D_u D_v + \frac{b}{4} e^{b\phi - U + W} E_u E_v \,. \tag{2.25}$$

Now we can define

$$X = a\phi - V \qquad Y = b\phi + W \qquad Z = \phi + \frac{al + (a+b)n}{4nl} V - \frac{bl + (a+b)m}{4ml} W \tag{2.26}$$

and in terms of (f, g) coordinates, the equations (2.18), (2.19), (2.23), (2.24), (2.25) are of the forms

$$\begin{aligned} (f+g)X_{fg} + \frac{1}{2}X_f + \frac{1}{2}X_g = \left(\frac{a^2}{4} + \frac{n(m+l)}{m+n+l}\right) e^X D_f D_g + \\ + \left(\frac{ab}{4} - \frac{mn)}{m+n+l}\right) e^Y E_f E_g \end{aligned} \tag{2.27}$$

$$(f+g)Y_{fg}+\frac{1}{2}Y_f+\frac{1}{2}Y_g = \left(\frac{ab}{4}-\frac{mn}{m+n+l}\right)e^X D_f D_g + \\ + \left(\frac{b^2}{4}+\frac{m(n+l))}{m+n+l}\right)e^Y E_f E_g \tag{2.28}$$

$$2D_{fg}+X_f D_g+X_g D_f = 0 \tag{2.29}$$

$$2E_{fg}+Y_f E_g+Y_g E_f = 0 \tag{2.30}$$

$$(f+g)Z_{fg}+\frac{1}{2}(Z_f+Z_g) = 0 \tag{2.31}$$

There exist a large class of solutions of the equation (2.31) [3, 17], which is of the form of the Euler-Darboux equation. We are not going to discuss all these solutions here. Instead, we just focus on the well-known Khan-Penrose-Szekeres solution:

$$Z=\kappa_1 \log\frac{w-p}{w+p}+\kappa_2\log\frac{r-q}{r+q}. \tag{2.32}$$

The equations (2.27), (2.28), (2.29), (2.30) are coupled differential equations, which may be taken as a generalized Ernst equations. We will make ansatz and solve these equations in the next section.

The equations (2.7), (2.8) can also be simplified to

$$S_f+(f+g)\left[\frac{V_f^2}{4n}+\frac{W_f^2}{4m}+\frac{(V_f-W_f)^2}{4l}+\phi_f^2\right]+\frac{1}{2}e^X D_f^2+\frac{1}{2}e^Y E_f^2 = 0 \tag{2.33}$$

$$S_g+(f+g)\left[\frac{V_g^2}{4n}+\frac{W_g^2}{4m}+\frac{(V_g-W_g)^2}{4l}+\phi_g^2\right]+\frac{1}{2}e^X D_g^2+\frac{1}{2}e^Y E_g^2 = 0 \tag{2.34}$$

by introducing

$$S=M-(1-\delta)\log(f+g)+\log(f_u g_v)+\frac{1}{2n}V-\frac{1}{2m}W \tag{2.35}$$

with

$$\delta=\frac{m+n}{4mn}\,. \tag{2.36}$$

Once we solve the X,Y,Z,D,E, we can use the relation

$$\phi = \frac{4mnlZ+m(al+(a+b)n)X+n(bl+(a+b)m)Y}{\alpha} \tag{2.37}$$

$$V = \frac{4mnl(aZ-X)+n(bl+(a+b)m)(aY-bX)}{\alpha} \tag{2.38}$$

$$W = \frac{4mnl(Y-bZ)+m(al+(a+b)n)(aY-bX)}{\alpha} \tag{2.39}$$

to get ϕ,V,W, where

$$\alpha=mn(a+b)^2+l(ma^2+nb^2+4mn)\,. \tag{2.40}$$

With ϕ,V,W, one could integrate (2.33), (2.34) to get S and then M so as to obtain all the metric components. At the first looking, the three relations on ϕ,V,W are quite involved and the integration on S seems to be a forbidden task. Nevertheless, with the solutions we will discuss in this paper, we succeed in getting the exact metric components.

JHEP12(2004)016

3. Physical CPW solutions

In the study of the collision of the gravitational plane waves, one usually divides the spacetime into four regions: past P-region($u < 0,\ v < 0$), right R-region($u > 0,\ v < 0$), left L-region($u < 0,\ v > 0$) and future F-region($u > 0,\ v > 0$), which describes the flat Minkowski spacetime, the incoming waves from right and left, and the colliding interaction region respectively. The general recipe to construct the CPW solutions is to solve the equations of motions in the forward region and then reduce the solutions to other regions, requiring the metric to be continuous and invertible in order to paste the solutions in different regions. More importantly, one need to impose the junction conditions to get an acceptable physical solution. In this section, we try to find the solutions to the equations of motions in the forward region and then reduce them to other regions and impose the junction conditions to get the physical CPW solutions.

3.1 Solutions to the equations of motions

As we mentioned, there exist a large class of solutions to the Euler-Darboux equation (2.31). And for the coupled differential equations (2.27)–(2.30), if we assume that $X_f = Y_f$, $X_g = Y_g$ and $D \propto E$, then the equations could reduce to the ones in 1-flux case [16], which are related to the Ernst equation. We will focus on two kinds of solutions of (2.27)–(2.30): one is $(pqrw)$-type and the other is $(f \pm g)$-type.

- $(pqrw)$-type (BS type) solution.

  Let us make the following ansatz:

$$X = -\log c_1 \frac{rw + pq}{rw - pq} \qquad Y = -\log c_2 \frac{rw + pq}{rw - pq} \tag{3.1}$$

$$D = \gamma_1 \cdot (pw - rq) \qquad E = \gamma_2 \cdot (pw - rq) \tag{3.2}$$

  where

$$p := \sqrt{\frac{1}{2} - f} \qquad q := \sqrt{\frac{1}{2} - g} \qquad r := \sqrt{\frac{1}{2} + f} \qquad w := \sqrt{\frac{1}{2} + g}\,. \tag{3.3}$$

  Note that there seem to be two free parameters c_1, c_2 in X, Y, however, the continuity condition on the metric in four patches restrict $c_1 = c_2 = 1$. And the equations (2.27), (2.28) require

$$\gamma_1^2 = 2 \cdot \frac{b(b-a)(m+n+l) + 4m(2n+l)}{\alpha} \tag{3.4}$$

$$\gamma_2^2 = 2 \cdot \frac{a(a-b)(m+n+l) + 4n(2m+l)}{\alpha} \tag{3.5}$$

  After a straightforward but tedious calculation, we obtain

$$S_f + (f+g)\left[\frac{4mnl}{\alpha} Z_f^2 + \frac{\gamma_1^2 + \gamma_2^2}{8} X_f^2\right] + \frac{1}{2} e^X D_f^2 + \frac{1}{2} e^Y E_f^2 = 0 \tag{3.6}$$

and the similar relation to S_g. After integration over f and g, one has

$$\begin{aligned} S =& b_1 \log(1-2f)(1+2g) + b_2 \log(1+2f)(1-2g) + \\ &+ b_3 \log(f+g) + b_4 \log\left(1+4fg+\sqrt{(1-4f^2)(1-4g^2)}\right) \end{aligned} \tag{3.7}$$

where

$$b_1 = \frac{\gamma_1^2+\gamma_2^2}{8} + \frac{4mnl\kappa_1^2}{\alpha} \tag{3.8}$$

$$b_2 = \frac{\gamma_1^2+\gamma_2^2}{8} + \frac{4mnl\kappa_2^2}{\alpha} \tag{3.9}$$

$$b_3 = -\frac{\gamma_1^2+\gamma_2^2}{8} - \frac{4mnl}{\alpha}(\kappa_1+\kappa_2)^2 \tag{3.10}$$

$$b_4 = \frac{8mnl\kappa_1\kappa_2}{\alpha} \tag{3.11}$$

and then the metric and the dilaton is given by

$$\begin{aligned} e^{-M} =& f_u g_v \left[(1-2f)(1+2g)\right]^{-b_1} \left[(1+2f)(1-2g)\right]^{-b_2} (f+g)^{-b_3-1+\delta} \times \\ &\times \left[1+4fg+4pqrw\right]^{-b_4} \left(\frac{rw+pq}{rw-pq}\right)^{s_1} \left[\left(\frac{w-p}{w+p}\right)^{\kappa_1} \left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{2l(ma+nb)}{\alpha}} \end{aligned} \tag{3.12}$$

$$e^{nA} = (f+g) \left[\left(\frac{w-p}{w+p}\right)^{\kappa_1} \left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{4mnla}{\alpha}} \left(\frac{rw+pq}{rw-pq}\right)^{s_2} \tag{3.13}$$

$$e^{mB} = (f+g) \left[\left(\frac{w-p}{w+p}\right)^{\kappa_1} \left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{4mnlb}{\alpha}} \left(\frac{rw+pq}{rw-pq}\right)^{s_3} \tag{3.14}$$

$$e^{lC} = \left[\left(\frac{w-p}{w+p}\right)^{\kappa_1} \left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{-4mnl(a+b)}{\alpha}} \left(\frac{rw+pq}{rw-pq}\right)^{s_4} \tag{3.15}$$

$$e^{\phi} = \left[\left(\frac{w-p}{w+p}\right)^{\kappa_1} \left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{4mnl}{\alpha}} \left(\frac{rw+pq}{rw-pq}\right)^{s_5} \tag{3.16}$$

where

$$s_1 = \frac{l(a-b)^2 + (b^2-a^2)(m-n) + 4l(m+n)}{2\alpha} \tag{3.17}$$

$$s_2 = \frac{n}{\alpha}[(b-a)(bl+(b+a)m)+4ml] \tag{3.18}$$

$$s_3 = \frac{m}{\alpha}[(a-b)(al+(b+a)n)+4nl] \tag{3.19}$$

$$s_4 = -\frac{l}{\alpha}[(a-b)(ma-nb)+8mn] \tag{3.20}$$

$$s_5 = -\frac{1}{\alpha}[l(ma+nb)+2mn(a+b)] \tag{3.21}$$

Therefore we have a two-parameter family of solution labelled by κ_1, κ_2.

- $(f \pm g)$-type solution.

As the flux-CPW solution with metric ansatz (1.2) discussed in [16, 19], we may assume that X, Y is the function of $(f+g)$ and D, E is the function of $(f-g)$. The equations (2.29), (2.30) tell us

$$D = \gamma_1 \cdot (f-g) \qquad E = \gamma_2 \cdot (f-g) \tag{3.22}$$

for some constants γ_1, γ_2. And (25), (26) can be solved by

$$X = -\log\left[\frac{(f+g)}{\alpha_1}\cosh^2\left(c_1 \log\frac{c_2}{f+g}\right)\right] \tag{3.23}$$

$$Y = -\log\left[\frac{(f+g)}{\alpha_2}\cosh^2\left(c_1 \log\frac{c_2}{f+g}\right)\right] \tag{3.24}$$

where

$$\alpha_1 = \frac{2c_1^2[b(b-a)(m+n+l)+4m(2n+l)]}{\alpha\gamma_1^2} \tag{3.25}$$

$$\alpha_2 = \frac{2c_1^2[a(a-b)(m+n+l)+4n(2m+l)]}{\alpha\gamma_2^2}\,. \tag{3.26}$$

After integrating f and g, one can read

$$\begin{aligned} S = {} & b_1 \log(1-2f)(1+2g) + b_2 \log(1+2f)(1-2g) + b_3 \log(f+g) + \\ & + b_4 \log(1+4fg+\sqrt{(1-4f^2)(1-4g^2)}) + b_5 \log\cosh\left(c_1\log\frac{c_2}{f+g}\right) \end{aligned} \tag{3.27}$$

where

$$b_1 = \frac{4mnl\kappa_1^2}{\alpha} \tag{3.28}$$

$$b_2 = \frac{4mnl\kappa_2^2}{\alpha} \tag{3.29}$$

$$b_3 = -\frac{(\alpha_1\gamma_1^2+\alpha_2\gamma_2^2)(4c_1^2+1)}{8c_1^2} - \frac{4mnl}{\alpha}(\kappa_1+\kappa_2)^2 \tag{3.30}$$

$$b_4 = \frac{8mnl\kappa_1\kappa_2}{\alpha} \tag{3.31}$$

$$b_5 = -\frac{\alpha_1\gamma_1^2+\alpha_2\gamma_2^2}{2c_1^2}\,. \tag{3.32}$$

And the metric components and the dilaton follow

$$\begin{aligned} e^{-M} = {} & t_0 f_u g_v \Big[(1-2f)(1+2g)\Big]^{-b_1}\Big[(1+2f)(1-2g)\Big]^{-b_2}(f+g)^{-b_3-1+\delta} \times \\ & \times\left[\cosh\left(c_1\log\frac{c_2}{f+g}\right)\right]^{-b_5}\left[(f+g)\cosh^2\left(c_1\log\frac{c_2}{f+g}\right)\right]^{s_1} \times \\ & \times\Big[1+4fg+4pqrw\Big]^{-b_4}\left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{-\frac{2l(ma+nb)}{\alpha}} \end{aligned} \tag{3.33}$$

$$e^{nA} = t_1(f+g)\left[(f+g)\cosh^2\left(c_1 \log\frac{c_2}{f+g}\right)\right]^{s_2}\left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{4mnla}{\alpha}} \quad (3.34)$$

$$e^{mB} = t_2(f+g)\left[(f+g)\cosh^2\left(c_1 \log\frac{c_2}{f+g}\right)\right]^{s_3}\left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{4mnlb}{\alpha}} \quad (3.35)$$

$$e^{lC} = t_1^{-1}t_2^{-1}\left[(f+g)\cosh^2\left(c_1 \log\frac{c_2}{f+g}\right)\right]^{s_4}\left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{-4mnl(a+b)}{\alpha}} \quad (3.36)$$

$$e^{\phi} = t_3\left[(f+g)\cosh^2\left(c_1 \log\frac{c_2}{f+g}\right)\right]^{s_5}\left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{4mnl}{\alpha}} \quad (3.37)$$

where s_i's are the same parameters above and

$$\begin{aligned} t_0 &= \alpha_1^{-\frac{4ml+b((b-a)l+(a+b)(m-n))}{2\alpha}}\alpha_2^{-\frac{4nl-a((b-a)l+(a+b)(m-n))}{2\alpha}} \\ t_1 &= \alpha_1^{-n\frac{4ml+b(bl+(a+b)m)}{\alpha}}\alpha_2^{na\frac{bl+(a+b)m}{\alpha}} \\ t_2 &= \alpha_1^{mb\frac{al+(a+b)n}{\alpha}}\alpha_2^{-m\frac{4nl+a(al+(a+b)n)}{\alpha}} \\ t_3 &= \alpha_1^{-m\frac{al+(a+b)n}{\alpha}}\alpha_2^{-n\frac{bl+(a+b)m}{\alpha}} \end{aligned}$$

After imposing the continuity condition on the metric, one get quite involved constraints on $c_1, c_2, \alpha_1, \alpha_2$. One can choose $c_2 = 0$ and simplify the constraints to be $\alpha_1 = \alpha_2 = 1$, which lead to $t_i = 1$ for $i = 0, \ldots, 3$ and the relations among c_1^2 and γ_1, γ_2. In short, we have a three-parameter family of solutions, labelled by κ_1, κ_2 and c_1.

Up to now, we have solved the equations of motions in the F-region. Actually one can reduce the above solutions to the ones for the L-region, the R-region, and the P-region if one do the following replacements:

$$f(u) = f_0 \quad f_u(1-2f)^{-b_1}\,|_{f=f_0} = -1 \qquad \text{for } u < 0 \quad (3.38)$$

$$g(v) = g_0 \quad g_v(1-2g)^{-b_2}\,|_{g=g_0} = -1 \qquad \text{for } v < 0 \quad (3.39)$$

where b_1, b_2 have been given in (3.8), (3.28), (3.9), (3.29) and f_0, g_0 are constants. Taking into account of the continuous and invertible conditions on the metric, we are able to fix the values of

$$f_0 = g_0 = \frac{1}{2}\,. \quad (3.40)$$

Actually, in order to simplify the expressions, we have use the continuous and invertible condition to fix the value of c_i's and α_i's.

The metric (1.3) is in the Rosen coordinates. In order to show that the incoming waves looks like the plane-waves, it is more convenient to write the metric in Brinkmann coordinates. In the incoming region, e.g. R-region ($v < 0$), the metric in the Brinkmann coordintes takes the form:

$$ds^2 = 2dx^+dx^- + \left(H_x(x^+)\sum_{i=1}^{n} X_i^2 + H_y(x^+)\sum_{j=1}^{m} Y_j^2 + H_z(x^+)\sum_{k=1}^{l} Z_k^2\right)(dx^+)^2 +$$

$$+\sum_{i=1}^{n} dX_i^2 + \sum_{j=1}^{m} dY_j^2 + \sum_{k=1}^{l} dZ_k^2\,, \tag{3.41}$$

where x^+ is related to u through

$$e^{-M}du = dx^+ \tag{3.42}$$

and

$$\begin{aligned} H_x &= e^{-A}\frac{d^2e^A}{dx^{+2}} = e^{2M}(A_{uu}+M_uA_u+A_u^2) \\ H_y &= e^{-B}\frac{d^2e^B}{dx^{+2}} = e^{2M}(B_{uu}+M_uB_u+B_u^2) \\ H_z &= e^{-C}\frac{d^2e^C}{dx^{+2}} = e^{2M}(C_{uu}+M_uC_u+C_u^2) \end{aligned}$$

It is straightforward to write down the explicit Brinkmann form of the metric corresponding to different CPW solutions.

3.2 Imposing the junction conditions

The junction conditions are essential to make the solutions obtained be physically acceptable. The detailed discussions on the junction conditions can be found in [16]. The key points are

1. The metric must be continuous and invertible;
2. If the metric is C^1, then impose the Lichnerowicz condition: the metric has to be at least C^2. Otherwise, if the metric is piecewise C^1, then impose the OS junction conditions [18] which require
$$g_{\mu\nu}\,, \qquad \sum_{ij} g^{ij}g_{ij,0}\,, \qquad \sum_{ij} g^{i0}g_{ij,0}\,, \qquad (i,j\neq 0)\,. \tag{3.43}$$
to be continuous across the null surface (note that "0" in the above formulae stands for $u=0$ or $v=0$). From our ansatz on the metric, the OS condition means that U,V,M need to be continuous and $U_u=0$ across the junction at $u=0$. The same happens at the junction $v=0$.
3. The curvature invariants R, R_2 do not blow up at the junction.

The first condition is natural to paste the solutions in different regions together. The second condition comes from the requirement that the stress tensor could be piecewise continuous instead of being continuous, namely the stress tensor may have finite jump but not δ-function jump across the junction, such that the Ricci tensor is allowed to be piecewise continuous. The third condition is from the requirement that the curvature invariants R and R_2 should not have poles at the junction.

In order to study the behavior of the solution near the junction, we assume that near junction,

$$f(u) = f_0(1-d_1u^{n_1}) \qquad u\sim 0^+ \tag{3.44}$$
$$g(v) = g_0(1-d_2v^{n_2}) \qquad v\sim 0^+ \tag{3.45}$$

where n_i, $i=1,2$ are the boundary exponents.

From the continuous condition on the metric and conditions (3.44), (3.45), one has

$$f_0 = g_0 = \frac{1}{2} \tag{3.46}$$

and

$$b_i = 1 - \frac{1}{n_i}\,, \quad d_i = \left(\frac{2}{n_i}\right)^{n_i}\,, \qquad i = 1, 2\,. \tag{3.47}$$

With the expansion (3.44), (3.45), it is not hard to figure out the near junction behavior of the metric components. Near $u \sim 0$ (same for $v \sim 0$), the most singular terms in the metric are:

$$M_u \sim \left(u^{\frac{n_1}{2}-1} e_0(\nu) + \text{l.s.t.}\right)\Theta(u) \tag{3.48}$$

$$A_u \sim \left(u^{\frac{n_1}{2}-1} e_1(\nu) + \text{l.s.t.}\right)\Theta(u) \tag{3.49}$$

$$B_u \sim \left(u^{\frac{n_1}{2}-1} e_2(\nu) + \text{l.s.t.}\right)\Theta(u) \tag{3.50}$$

$$C_u \sim \left(u^{\frac{n_1}{2}-1} e_3(\nu) + \text{l.s.t.}\right)\Theta(u) \tag{3.51}$$

where $e_i(v), i = 0, \ldots, 3$ are some nonzero functions of v and $\Theta(u)$ is the Heaviside step function.[1] After imposing the condition 2 above, we have

$$\begin{cases} 1.\ \ 1 < n_i \leq 2 & \text{metric is piecewise } C^1 \\ 2.\ \ n_i > 2 & \text{metric is at least piecewise } C^2 \end{cases} \tag{3.52}$$

As the last step, we have to impose the condition 3 on R, R_2. Now, one has

$$R = 2e^M \phi_u \phi_v + \frac{m-n+l}{m+n+l} D_u D_v \frac{e^{M+X}}{f+g} + \frac{n-m+l}{m+n+l} E_u E_v \frac{e^{M+Y}}{f+g} \tag{3.53}$$

from the Einstein equations and also

$$R_2 = 2e^{2M} R_{uv}^2 + 2e^{2M} R_{uu} R_{vv} + ne^{-2A} R_{xx}^2 + me^{-2B} R_{yy}^2 + le^{-2C} R_{zz}^2\,. \tag{3.54}$$

From the near junction behavior of the metric and fields, one finds that near $u \sim 0$ (same for $v \sim 0$),

$$R \sim u^{\frac{n_1}{2}-1} \sim R_2 \tag{3.55}$$

In order to keep R, R_2 from blowing up, one needs to ask $n_i \geq 2$.

Taking into account the constraints from all the junction conditions, we have the following physical possibilities :

$$b_i = 1 - \frac{1}{n_i} \tag{3.56}$$

and

$$\begin{cases} 1.\ \ n_i = 2 & \text{metric is piecewise } C^1 \\ 2.\ \ n_i > 2 & \text{metric is at least piecewise } C^2 \end{cases} \tag{3.57}$$

[1] Actually, the detailed discussions on near junction behavior are quite similar to the ones in [16, 19] so we just give out the final answer to save the space.

JHEP12(2004)016

This leads to the following relations

$$\frac{1}{2} \leq b_i \leq 1\,, \qquad i = 1, 2\,. \tag{3.58}$$

Let us turn to our solutions in the above section. From (3.28), (3.29), it is obvious that we have physical acceptable solutions since there are two free parameters κ_1, κ_2. Therefore, physical $(f \pm g)$ solutions always exist. As for the $(pqrw)$ type solution, from (3.8), (3.9), the only trouble may comes from $b_i > 1$, which could happen when

$$\frac{\gamma_1^2 + \gamma_2^2}{8} > 1\,. \tag{3.59}$$

This is possible. One example is when $a = -b$ and $l = 1$, which lead to

$$\frac{\gamma_1^2 + \gamma_2^2}{8} = \frac{a^2(m+n+1) + 4mn + m + n}{a^2(m+n) + 4mn}\,. \tag{3.60}$$

Since m, n are integers, the above relation must be bigger than 1. It seems that there may not exist physically acceptable $(pqrw)$-type solution. Certainly, a case-by-case study is needed. One can hope that in general, there could exist $(pqrw)$ type solutions.

Next, we would like to consider the future singularity issue. In the various four-dimensional CPW solutions and higher-dimensional CPW solutions, it has been found that the future singularity is always developed [4, 5, 16, 19]. We now show that it is the case here. Define a hyper-surface S_0:

$$f(u) + g(v) = 0 \tag{3.61}$$

near which the metric may blow up or vanish. The singular behavior of the metric near S_0 is easy to read out. From the fact that

$$e^{lC} = e^{2U - nA - mB} = (f+g)^2 e^{-nA} e^{-mB}\,, \tag{3.62}$$

we know that it is impossible to require the metric to be regular and invertible near S_0. In other words, the metric must be singular at S_0. To see that this singularity is not a coordinate singularity, let us check the curvature invariants R, R_2, R_4, whose most singular terms near S_0 all take the form:

$$e^{2M}(f+g)^{-4} \sim (f+g)^{2\beta} \tag{3.63}$$

where

$$\beta = b_3 - 1 - \delta - s_1(1 - 2(1-\epsilon)c_1) - b_5 c_1 - (\kappa_1 + \kappa_2)\frac{2l(ma+nb)}{\alpha}\,. \tag{3.64}$$

For the $(pqrw)$-type solution, with some efforts, one can get

$$\beta \leq -1 - \frac{1}{\alpha}(na^2 + mb^2 + l(a-b)^2 + (4mn + 3ml + 3nl)) < 0 \tag{3.65}$$

so the future curvature singularity will be developed. As for the $(f \pm g)$-type solution, the analysis on β is quite involved but we believe that the same conclusion could be reached.

4. Some special cases

In the above sections, we have worked on a quite general higher dimensional gravity theory with two fluxes and dilaton. The key point in this paper is the metric ansatz (1.3). One interesting issue is to revisit the flux-CPW solutions in various special cases, with the metric (1.3). We will find that the CPW solutions in some cases could be reduced directly from the solutions we have obtained in the above section, while in other cases, we have to solve the reduced equations of motions independently.

4.1 Case I: one-flux CPW solutions

The first case is to set $b = 0, E = 0$ such that our theory reduce to the one-flux dilatonic gravity, which has been discussed thoroughly in [16] under a different metric ansatz (1.2). In [11], the CPW solutions of the four-dimensional Einstein-Maxwell-dilaton theory has been constructed. In our higher dimensional case, we have

$$\begin{aligned} X &= a\phi - V \\ Y &= W \\ Z &= \phi + \frac{a}{4nl}((l+n)V - nW)\,. \end{aligned}$$

The equations on X, Y, Z, D are easily reduced from (2.27), (2.28), (2.31), (2.29). It is easy to find out the solutions to X, D, Z. However, in order to solve Y, one needs to introduce

$$K = \frac{mn}{m+n+l}X + \left(\frac{a^2}{4} + \frac{n(m+l)}{m+n+l}\right)Y \tag{4.1}$$

which satisfy

$$(f+g)K_{fg} + \frac{1}{2}(K_f + K_g) = 0 \tag{4.2}$$

with the solution chosen to be

$$K = \kappa_3 \log\frac{w-p}{w+p} + \kappa_4 \log\frac{r-q}{r+q}\,. \tag{4.3}$$

Therefore, from the solutions K, X, one can get the solutions of Y.

In terms of Z, K, X, we get the equation on S_f(same to S_g)

$$\begin{aligned} 0 = S_f + (f+g)\Bigg[&\frac{4nlZ_f^2}{4ln + a^2(l+n)} + \frac{(m+n+l)X_f^2}{a^2(m+n+l) + 4n(m+l)} + \\ &+ \frac{4(n+m+l)^2K_f^2}{[4ln + a^2(l+n)][a^2(m+n+l)+4n(m+l)]m}\Bigg] + \frac{1}{2}e^X D_f^2\,. \end{aligned} \tag{4.4}$$

In this case, the solutions to X, D could be $(pqrw)$-type as in (3.1), (3.2) with

$$c_1 = 1\,, \qquad \gamma_1^2 = \frac{8(m+n+l)}{a^2(m+n+l) + 4n(m+l)} \tag{4.5}$$

or $(f \pm g)$-type as in (3.23), (3.22) with

$$\frac{\gamma_1^2}{c_1^2} = \frac{8(m+n+l)}{a^2(m+n+l)+4n(m+l)} \,. \tag{4.6}$$

Then in a straightforward way we get the same form of S as (3.7), (3.27) but with

$$\begin{aligned}
b_1 &= \frac{4nlm[a^2(m+n+l)+4n(m+l)]\kappa_1^2 + 4(n+m+l)^2\kappa_3^2 + \epsilon m(m+n+l)[4ln+a^2(l+n)]}{[4ln+a^2(l+n)][a^2(m+n+l)+4n(m+l)]m} \\
b_2 &= \frac{4nlm[a^2(m+n+l)+4n(m+l)]\kappa_2^2 + 4(n+m+l)^2\kappa_4^2 + \epsilon m(m+n+l)[4ln+a^2(l+n)]}{[4ln+a^2(l+n)][a^2(m+n+l)+4n(m+l)]m} \\
b_3 &= -\frac{4nl(\kappa_1+\kappa_2)^2}{4ln+a^2(l+n)} - \frac{4(n+m+l)^2(\kappa_3+\kappa_4)^2}{[4ln+a^2(l+n)][a^2(m+n+l)+4n(m+l)]m} - \\
&\quad -\frac{(m+n+l)(4(1-\epsilon)c_1^2+1)}{a^2(m+n+l)+4n(m+l)} \\
b_4 &= \frac{8nlm[a^2(m+n+l)+4n(m+l)]\kappa_1\kappa_2 + 8(n+m+l)^2\kappa_3\kappa_4}{[4ln+a^2(l+n)][a^2(m+n+l)+4n(m+l)]m} \\
b_5 &= (\epsilon-1)\frac{4(m+n+l)}{a^2(m+n+l)+4n(m+l)}
\end{aligned} \tag{4.7}$$

Here

$$\epsilon = \begin{cases} 1, & \text{for } (pqrw)\text{-type solution} \\ 0, & \text{for } (f \pm g)\text{-type solution} \end{cases} \tag{4.8}$$

Similarly, we have metric components and dilaton field as

$$\begin{aligned}
e^{-M} &= f_u g_v \left[(1-2f)(1+2g)\right]^{-b_1}\left[(1+2f)(1-2g)\right]^{-b_2}(f+g)^{-b_3-1+\delta} \times \\
&\quad \times \left[1+4fg+4pqrw\right]^{-b_4}\left[\cosh\left(c_1 \log\frac{c_2}{f+g}\right)\right]^{-b_5} \times \\
&\quad \times e^{-s_1X+a_1K} \exp\left\{\frac{2laZ}{4ln+a^2(l+n)}\right\}
\end{aligned} \tag{4.9}$$

$$e^{nA} = (f+g)e^{-s_2X+a_2K}\exp\left\{\frac{4nlaZ}{4ln+a^2(l+n)}\right\} \tag{4.10}$$

$$e^{mB} = (f+g)e^{-s_3X+a_3K} \tag{4.11}$$

$$e^{lC} = e^{-s_4X+a_4K}\exp\left\{\frac{-4nlaZ}{4ln+a^2(l+n)}\right\} \tag{4.12}$$

$$e^{\phi} = e^{-s_5X+a_5K}\exp\left\{\frac{4nlZ}{4ln+a^2(l+n)}\right\} \tag{4.13}$$

where

$$s_1 = \frac{2(m-n+l)}{a^2(m+n+l)+4n(m+l)}, \qquad a_1 = -\frac{2(m+n+l)[(m-l-n)a^2-4ln]}{[4ln+a^2(l+n)][a^2(m+n+l)+4n(m+l)]m} \tag{4.14}$$

$$s_2 = \frac{4n(m+l)}{a^2(m+n+l)+4n(m+l)}, \qquad a_2 = \frac{4(m+n+l)a^2n}{[4ln+a^2(l+n)][a^2(m+n+l)+4n(m+l)]} \tag{4.15}$$

$$s_3 = \frac{-4mn}{a^2(m+n+l)+4n(m+l)}, \qquad a_3 = -\frac{4(m+n+l)}{a^2(m+n+l)+4n(m+l)} \tag{4.16}$$

$$s_4 = \frac{-4nl}{a^2(m+n+l)+4n(m+l)}, \qquad a_4 = \frac{4(m+n+l)l(a^2+4n)}{[4ln+a^2(l+n)][a^2(m+n+l)+4n(m+l)]} \tag{4.17}$$

$$s_5 = -\frac{a(m+n+l)}{a^2(m+n+l)+4n(m+l)}, \qquad a_5 = \frac{4an(m+n+l)}{[4ln+a^2(l+n)][a^2(m+n+l)+4n(m+l)]} \tag{4.18}$$

and

$$e^{-X} = \begin{cases} \frac{rw+pq}{rw-pq}, & \text{for } (pqrw)\text{-type solution} \\ (f+g)\cosh^2\left(c_1 \log \frac{c_2}{f+g}\right) & \text{for } (f\pm g)\text{-type solution} \end{cases} \tag{4.19}$$

and K is of the form (4.3), Z is of the form (2.32).

In short, we have a four-parameter family of the $(pqrw)$-type solutions labelled by κ_1, κ_2, κ_3, κ_4 and a five-parameter family of the $(f \pm g)$-type solutions labelled by κ_1, κ_2, κ_3, κ_4 and c_1.

In order to get the physical solutions, we impose the junction conditions which similarly leads to $n_i \geq 2$, or equivalently, $1 \geq b_i \geq 1/2$, for $i = 1, 2$. As before, the $(f \pm g)$-type solution exist anyway. The only trouble in $(pqrw)$-type solutions may come from

$$\frac{m+n+l}{a^2(m+n+l)+4n(m+l)} > 1\,. \tag{4.20}$$

It is not difficult to see that this is impossible since the l.h.s. of the above relation is always less than 1. Therefore, in the one-flux case, we can always have two classes of dilatonic flux-CPW solutions.

In the same way, one can study the future singularity of these solutions. Near S_0 : $f + g = 0$, the metric is always singular. And the most singular terms in the curvature invariants are all of the form $(f + g)^{2\beta}$ with

$$\beta = b_3 - 1 - \delta - s_1(1 - 2(1-\epsilon)c_1) - b_5 c_1 - (\kappa_1 + \kappa_2)\frac{2la}{4ln + a^2(l+n)}\,. \tag{4.21}$$

We have proved that β is negative for both types of solutions. This indicates that the future curvature singularity is unavoidable.

Furthermore, if one turn off the dilaton, one gets a very interesting case. Now we have $a = b = 0, \phi = 0, E = 0$ and so $Z = 0$. The reduction of the equations of motions is simple: just let $a = 0$ and $Z = 0$ in the above discussions. And the solutions of the equations and the metric components could be obtained by setting $\kappa_1 = \kappa_2 = 0$ and $a = 0$. We will omit the details here. The answer looks simpler. Imposing the junction conditions lead to the same conclusion, namely we have well-defined physical pure-flux-CPW solutions: a two-parameter family of the $(pqrw)$-type solutions labelled by κ_3, κ_4 and a three-parameter family of the $(f \pm g)$-type solutions labelled by κ_3, κ_4 and c_1. Similarly, the discussion on the future singularity follows directly and reach the same conclusion.

There are two remarkable solutions in the pure flux case without dilaton. Firstly, we can have physical CPW solutions in the higher-dimensional Maxwell-Einstein gravity. Now the flux is a two form, being the field strength of the U(1) gauge field. So $n = 1$ but we have no constraints on the number m, l of other dimensions. Therefore with metric (1.3), there exist not only the physically well-defined higher-dimensional Bell-Szekeres $(pqrw)$-type solutions (see also [13]) but also well-defined $(f \pm g)$-type solutions. When $m + l = 1$, the $(pqrw)$-type solution reduce to the well-known Bell-Szekeres solution in four dimension. Secondly, we can have flux-CPW solutions in the eleven-dimensional supergravity theory which has no dilaton. This corresponds to $n = 2, m + l = 7$. In [16], with the metric ansatz (1.2), we noticed that the above two flux-CPW solutions were impossible and the only pure flux-CPW solutions was four-dimensional BS solution. Now we show that with the metric ansatz (1.3), the higher-dimensional flux-CPW solutions without dilaton are possible. This indicate that in higher-dimensional gravity theory, the CPW solutions have more rich structure.

4.2 Pure dilatonic CPW solutions

If we set $a = b = 0, D = E = 0$, we have the pure dilatonic gravity theory. After the reduction of the equations of motions, we have $X = -V, Y = W$ and $Z = \phi$ and three decoupled equations

$$\begin{aligned}(f+g)X_{fg} + \frac{1}{2}(X_f + X_g) &= 0\\ (f+g)Y_{fg} + \frac{1}{2}(Y_f + Y_g) &= 0\\ (f+g)Z_{fg} + \frac{1}{2}(Z_f + Z_g) &= 0\,.\end{aligned}$$

For simplicity, we choose the solutions to the above equations to be of the Khan-Penrose-Szekeres type, i.e.

$$\begin{aligned}X &= \kappa_3 \log\frac{w-p}{w+p} + \kappa_4 \log\frac{r-q}{r+q}\,,\\ Y &= \kappa_5 \log\frac{w-p}{w+p} + \kappa_6 \log\frac{r-q}{r+q}\,,\\ Z &= \kappa_1 \log\frac{w-p}{w+p} + \kappa_2 \log\frac{r-q}{r+q}\,.\end{aligned}$$

Then the metric components and the dilaton field are of the forms:

$$\begin{aligned}e^{-M} &= f_u g_v \left[(1-2f)(1+2g)\right]^{-b_1} \left[(1+2f)(1-2g)\right]^{-b_2} (f+g)^{-b_3-1+\delta} \times\\ &\quad \times \left[1 + 4fg + 4pqrw\right]^{-b_4} \exp\left\{-\frac{X}{2n} - \frac{Y}{2m}\right\} \qquad (4.22)\\ e^{nA} &= (f+g)e^{-X} \qquad (4.23)\\ e^{mB} &= (f+g)e^{-Y} \qquad (4.24)\\ e^{lC} &= e^{X+Y} \qquad (4.25)\\ e^{\phi} &= e^{Z} \qquad (4.26)\end{aligned}$$

where

$$\begin{aligned}b_1 &= \frac{\kappa_3^2}{4n} + \frac{\kappa_5^2}{4m} + \frac{(\kappa_3+\kappa_5)^2}{4l} + \kappa_1^2 \qquad (4.27)\\ b_2 &= \frac{\kappa_4^2}{4n} + \frac{\kappa_6^2}{4m} + \frac{(\kappa_4+\kappa_6)^2}{4l} + \kappa_2^2 \qquad (4.28)\\ b_3 &= -\left(\frac{(\kappa_3+\kappa_4)^2}{4n} + \frac{(\kappa_5+\kappa_6)^2}{4m} + \frac{(\kappa_3+\kappa_4+\kappa_5+\kappa_6)^2}{4l} + (\kappa_1+\kappa_2)^2\right) \qquad (4.29)\\ b_4 &= \frac{\kappa_3\kappa_4}{2n} + \frac{\kappa_5\kappa_6}{2m} + \frac{(\kappa_3+\kappa_5)(\kappa_4+\kappa_6)}{2l} + 2\kappa_1\kappa_2\,. \qquad (4.30)\end{aligned}$$

Now we have a six-parameter family of the solutions labelled by κ_i, $i = 1, \ldots, 6$.

As before, the junction conditions require that $1 \geq b_i \geq 1/2, i = 1, 2$, which could be easily achieved. Also the metric is singular at S_0. And the curvature invariants have singular terms $(f + g)^{2\beta}$ near S_0, where

$$\begin{aligned}\beta &= b_3 - 1 - \delta + \frac{\kappa_3 + \kappa_4}{2n} + \frac{\kappa_5 + \kappa_6}{2m} \\ &= -1 - \frac{1}{4n}(\kappa_3 + \kappa_4 - 1)^2 - \frac{1}{4m}(\kappa_5 + \kappa_6 - 1)^2 < 0\end{aligned} \tag{4.31}$$

Therefore, it is impossible for the solution to avoid the singularity in the future.

In this case, if we set $\phi = 0$, we have the pure gravitational CPW solution, which could be reduced from the above solutions in a direct way. In [12], the higher even dimensional pure CPW solutions have been constructed. The solutions take a quite similar form as (4.22). Our solutions exist in any higher dimensions.

4.3 Two-flux CPW solutions without dilaton

If we set $a = b = 0, \phi = 0$ such that the dilaton field is turned off, we obtain a gravity theory without the dilaton. Now we have $X = -V, Y = W$ and $Z = 0$. The equations on X, Y, D, E take the same form. Therefore, the solutions could be read from (3.12)–(3.15) and (3.33)–(3.36) directly by setting $a = b = 0, \kappa_1 = \kappa_2 = 0$. The imposition of the junction conditions and the discussion of the future singularity follow in a straightforward way.

5. Discussions and conclusion

In this paper, starting from the metric (1.3), we tried to construct the colliding plane wave solutions in a higher dimensional gravity theory with the dilaton and the fluxes. We find that in general we have two classes of well-defined physical solutions after imposing the junction conditions. Of particular interest is the one-flux case without the dilaton field. In this case, we obtain the higher dimensional Bell-Szekeres solutions in Maxwell-Einstein gravity and also flux-CPW solutions in 11D supergravity theory. Our discovery of the well-defined pure flux-CPW solutions in higher dimension indicate that the flux-CPW solutions is more intricate since there are more degrees of freedom in the metric. One may try to find more CPW solutions from a more generic metric ansatz.

One interesting point is that in our discussion we limit ourselves to two restricted classes of solutions. It must be possible to consider other kinds of solutions in our context.

Another point is that the initial background is important to the formation of the singularity. So naively one can try to study the CPW solutions in AdS or dS spacetime. In the Anti-de Sitter spacetime, the study may shed light on the AdS/CFT correspondence. And in the de Sitter spacetime, the study may have some cosmological implications, if there do exist a positive tiny cosmological constant.

Acknowledgments

This work was supported by a grant of Chinese Academy of Science.

A. Riemann and Ricci tensors

With the metric ansatz (1.3), We have Ricci tensor

$$R_{uu} = -\frac{1}{2}\left[nA_{uu} + mB_{uu} + lC_{uu} + nM_uA_u + mM_uB_u + lM_uC_u + \frac{1}{2}(nA_u^2 + mB_u^2 + lC_u^2)\right] \tag{A.1}$$

$$R_{vv} = -\frac{1}{2}\left[nA_{vv} + mB_{vv} + lC_{vv} + nM_vA_v + mM_vB_v + lM_vC_v + \frac{1}{2}(nA_v^2 + mB_v^2 + lC_v^2)\right] \tag{A.2}$$

$$R_{uv} = M_{uv} - \frac{n}{2}A_{uv} - \frac{m}{2}B_{uv} - \frac{l}{2}C_{uv} - \frac{1}{4}(nA_uA_v + mB_uB_v + lC_uC_v) \tag{A.3}$$

$$R_{xx} = -\frac{1}{2}e^{M+A}\left[2A_{uv} + nA_uA_v + \frac{m}{2}(A_uB_v + A_vB_u) + \frac{l}{2}(A_uC_v + A_vC_u)\right] \tag{A.4}$$

$$R_{yy} = -\frac{1}{2}e^{M+B}\left[2B_{uv} + mB_uB_v + \frac{n}{2}(A_uB_v + A_vB_u) + \frac{l}{2}(B_uC_v + B_vC_u)\right] \tag{A.5}$$

$$R_{zz} = -\frac{1}{2}e^{M+C}\left[2C_{uv} + lB_uB_v + \frac{n}{2}(A_uC_v + A_vC_u) + \frac{m}{2}(B_uC_v + B_vC_u)\right] \tag{A.6}$$

where $x = x_i$ with $i = 1, \ldots, n$, $y = y_j$ with $j = 1, \ldots, m$ and $z = z_k$ with $k = 1, \ldots, l$. And also we have the independent non-vanishing components of the Riemann tensor as following:

$$R_{uvuv} = -e^M M_{uv} \tag{A.7}$$

$$R_{uxvx} = -e^A\left(\frac{1}{2}A_{uv} + \frac{1}{4}A_uA_v\right) \tag{A.8}$$

$$R_{vxvx} = -e^A\left(\frac{1}{2}A_{vv} + \frac{1}{2}M_vA_v + \frac{1}{4}A_v^2\right) \tag{A.9}$$

$$R_{uxux} = -e^A\left(\frac{1}{2}A_{uu} + \frac{1}{2}M_uA_u + \frac{1}{4}A_u^2\right) \tag{A.10}$$

$$R_{uyvy} = -e^B\left(\frac{1}{2}B_{uv} + \frac{1}{4}B_uB_v\right) \tag{A.11}$$

$$R_{vyvy} = -e^B\left(\frac{1}{2}B_{vv} + \frac{1}{2}M_vB_v + \frac{1}{4}B_v^2\right) \tag{A.12}$$

$$R_{uyuy} = -e^B\left(\frac{1}{2}B_{uu} + \frac{1}{2}M_uB_u + \frac{1}{4}B_u^2\right) \tag{A.13}$$

$$R_{uzvz} = -e^C\left(\frac{1}{2}C_{uv} + \frac{1}{4}C_uC_v\right) \tag{A.14}$$

$$R_{uzuz} = -e^C\left(\frac{1}{2}C_{uu} + \frac{1}{2}M_uC_u + \frac{1}{4}C_u^2\right) \tag{A.15}$$

$$R_{vzvz} = -e^C\left(\frac{1}{2}C_{vv} + \frac{1}{2}M_vC_v + \frac{1}{4}C_v^2\right) \tag{A.16}$$

$$R_{xyxy} = -\frac{1}{4}e^{M+A+B}(A_uB_v + A_vB_u) \tag{A.17}$$

$$R_{xzxz} = -\frac{1}{4}e^{M+A+C}(A_uC_v + A_vC_u) \tag{A.18}$$

JHEP12(2004)016

$$R_{yzyz} = -\frac{1}{4}e^{M+B+C}(C_uB_v + C_vB_u) \tag{A.19}$$

$$R_{x_ix_{i'}x_ix_{i'}} = -\frac{1}{2}e^{M+2A}A_uA_v, \qquad (i \neq i') \tag{A.20}$$

$$R_{y_jy_{j'}y_jy_{j'}} = -\frac{1}{2}e^{M+2B}B_uB_v, \qquad (j \neq j') \tag{A.21}$$

$$R_{z_kz_{k'}z_kz_{k'}} = -\frac{1}{2}e^{M+2C}C_uC_v, \qquad (k \neq k')\,. \tag{A.22}$$

Also, we have

$$\begin{aligned} R_4 = \frac{e^{2M}}{4}\Big(&16M_{uv}^2 + 4n(A_{uu}A_v^2 + A_{vv}A_u^2 + M_uA_uA_v^2 + M_vA_vA_u^2) + \\ &+ 4m(B_{uu}B_v^2 + B_{vv}B_u^2 + M_uB_uB_v^2 + M_vB_vB_u^2) + \\ &+ 4l(C_{uu}C_v^2 + C_{vv}C_u^2 + M_uC_uC_v^2 + M_vC_vC_u^2) + \\ &+ 2nlA_uC_vA_vC_u + 2mnA_uB_vA_vB_u + 2mlB_uC_vB_vC_u + \\ &+ 2m(m+1)B_u^2B_v^2 + 2n(n+1)A_u^2A_v^2 + 2l(l+1)C_u^2C_v^2 + \\ &+ 8n(A_{uv}A_uA_v + A_{uv}^2 + A_{uu}A_{vv} + M_uA_uA_{vv} + M_vA_vA_{uu} + M_uA_uM_vA_v) + \\ &+ 8m(B_{uv}B_uB_v + B_{uv}^2 + B_{uu}B_{vv} + M_uB_uB_{vv} + B_vB_vB_{uu} + M_uB_uM_vB_v) + \\ &+ 8l(C_{uv}C_uC_v + C_{uv}^2 + C_{uu}C_{vv} + M_uC_uC_{vv} + C_vC_vC_{uu} + M_uC_uM_vC_v) + \\ &+ nm(A_v^2B_u^2 + B_v^2A_u^2) + nl(A_v^2C_u^2 + C_v^2A_u^2) + ml(C_v^2B_u^2 + B_v^2C_u^2)\Big)\,. \end{aligned} \tag{A.23}$$

References

[1] P.Szekeres, *Colliding gravitational waves*, *Nature* **228** (1970) 1183.

[2] K.A. Khan and R. Penrose, *Scattering of two impulsive gravitational waves*, *Nature* **229** (1971) 185.

[3] J.B. Griffiths, *Colliding plane waves in general relativity*, Clarendon Press, Oxford 1991.

[4] F.J. Tipler, *Singularities from colliding plane gravitational waves*, *Phys. Rev.* **D 22** (1980) 2929.

[5] U. Yurtsever, *Instability of Killing cauchy horizons in plane symmetric space-times*, *Phys. Rev.* **D 36** (1987) 1662; *Structure of the singularities produced by colliding plane waves*, *Phys. Rev.* **D 38** (1988) 1706.

[6] J. Centrella and R.A. Matzner, *Plane-symmetric cosmologies*, *Astrophys. J.* **230** (1979) 311; *Colliding gravitational waves in expanding cosmologies*, *Phys. Rev.* **D 25** (1982) 930.

[7] A. Feinstein, K.E. Kunze and M.A. Vazquez-Mozo, *Initial conditions and the structure of the singularity in pre-big-bang cosmology*, *Class. and Quant. Grav.* **17** (2000) 3599 [hep-th/0002070]; *Entropy generation and inflation in wave collision induced pre-big-bang cosmology*, *Phys. Lett.* **B 491** (2000) 190 [hep-th/0004094].

[8] V. Bozza and G. Veneziano, *$O(d,d)$-invariant collapse/inflation from colliding superstring waves*, *J. High Energy Phys.* **10** (2000) 035 [hep-th/0007159].

[9] H. Kodama, *Uniqueness and stability of higher-dimensional black holes*, hep-th/0403030.

[10] R. Emparan and H.S. Reall, *A rotating black ring in five dimensions*, *Phys. Rev. Lett.* **88** (2002) 101101 [hep-th/0110260].

[11] M. Gurses and E. Sermutlu, *Colliding gravitational plane waves in dilaton gravity*, *Phys. Rev.* **D 52** (1995) 809 [hep-th/9503218].

[12] M. Gurses, E.O. Kahya and A. Karasu, *Higher dimensional colliding gravitational plane wave metrics*, *Phys. Rev.* **D 66** (2002) 024029 [gr-qc/0204041];
M. Gurses and A. Karasu, *Some higher dimensional vacuum solutions*, *Class. and Quant. Grav.* **18** (2001) 509 [gr-qc/0007068].

[13] M. Gurses, Y. Ipekoglu, A. Karasu and C. Senturk, *Higher dimensional Bell-Szekeres metric*, *Phys. Rev.* **D 68** (2003) 084007 [gr-qc/0305105].

[14] M. Gutperle and B. Pioline, *Type IIB colliding plane waves*, *J. High Energy Phys.* **09** (2003) 061 [hep-th/0308167].

[15] P. Bell and P. Szekeres, *Interacting electromagnetic shock waves in general relativity*, *Gen. Rel. Grav.* **5** (1974) 275.

[16] B. Chen, C.-S. Chu, K. Furuta and F.-L. Lin, *Colliding plane waves in string theory*, *J. High Energy Phys.* **02** (2004) 020 [hep-th/0311135].

[17] W.B. Bonnor, J.B. Griffiths and M. MacCallum, *Physical interpretation of vacuum solutions of Einstein's equations. Part II. Time-dependent solutions*, *Gen. Rel. Grav.* **26** (1994) 687.

[18] S. O'Brein and J.L.Synge, *Jump conditions at discontinuities in general relativity*, *Comm. Dublin Inst. Adv. Stud.* **A9** (1952)

[19] B. Chen and J.-F. Zhang, *Two-flux colliding plane waves in string theory*, hep-th/0408009.